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GROUPS OF ELEMENTARY TRIGONOMETRY.

By G. A. MILLER.

In a preceding number of this journal we considered a substitution group of degree 6 and order 12, which is very useful in deriving formulas from each other.* In the present note we shall consider some groups of operations, which are associated with the triangle and the trigonometric functions. Nearly all of these groups are too elementary to exhibit many group properties, yet it is hoped that their formal statement will exhibit some important points in a clear light.

We begin with the operations upon the angle α which transform $\sin\alpha$ into $\pm\sin\alpha$. By reflecting the point representing α on the y-axis† we obtain a point representing an angle whose sine is equal to $\sin\alpha$; by reflecting the same point on the x-axis there results an angle whose sine is equal to $-\sin\alpha$. These two reflections, in succession, are equivalent to a rotation through the angle π, which transforms α into another angle whose sine is equal to $-\sin\alpha$. It is well known that the three given operations together with the identity constitute the four-group, or the axial group. Hence this axial group transforms the angle α so that $\sin\alpha$ goes into $\pm\sin\alpha$. It is clear that the same operations transform each of the other five trigonometric functions into ± the same named function.

By reflecting on the bisector of the first quadrant, every angle is changed into its complement. Hence the direct functions of α will be transformed into the co-functions by this operation. The group generated by this operation and the preceding axial group is the octic group, or the group of movements of the square. Hence this octic group transforms α in such a way that each function of α will go into ± itself or into ± its co-function. Since subtracting α from any angle β is equivalent to reflecting α on the diameter through $\beta/2$, it follows that the given operations result from taking the complement and supplement of the angle α, these operations being performed in any order. Hence by taking the complement or the supplement of an angle any number of times and in any order, we transform any of its six functions either into ± itself or into ± its co-function.

Now angles β_1, β_2 can be so selected that the two operations of subtracting α from them generate a dihedral rotation group of any even order, since the order of this group is $2\pi/d$, where d is the highest common factor of β_1, β_2, and 2π.‡ Theoretically, these groups are just as interesting as the group formed by taking the complement and supplement of the angle α, but they do not have such important applications in the solution of the triangle and hence need not be considered here.

In the preceding cases the operations of the group were performed upon the angle α. Instead of this, the operations may be performed upon functions of α. Perhaps the most interesting case results when the operations of inversion and subtracting from unity are performed upon $\sin^2\alpha$. The six operations of the group generated by the two given operations* transform $\sin^2\alpha$ into the following functions of $\sin^2\alpha$:

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*Vol. 5 (1898), p. 102.

†The angles are supposed to be represented by points on the unit circle.

‡Cf. Groups of Subtraction, MONTHLY, Nov., 1904.

$$\sin^2\alpha,\quad 1-\sin^2\alpha,\quad \frac{1}{\sin^2\alpha},\quad \frac{\sin^2\alpha-1}{\sin^2\alpha},\quad \frac{\sin^2\alpha}{\sin^2\alpha-1},\quad \frac{-1}{\sin^2\alpha-1}.$$

These are respectively,

$$\sin^2\alpha,\quad \cos^2\alpha,\quad \csc^2\alpha,\quad -\cot^2\alpha,\quad -\tan^2\alpha,\quad \sec^2\alpha.$$

Each of these functions is transformed into all the others by the operations of this group.

A more elementary example presents itself if it is observed that the four-group is the group of division and changing sign. In particular, it is the group of inversion and changing sign. Hence the three sets of four functions ($\pm\sin\alpha$, $\pm\csc\alpha$; $\pm\tan\alpha$, $\pm\cot\alpha$; $\pm\sec\alpha$, $\pm\csc\alpha$) are transformed among themselves by the operations of this group.

If the three sides of a triangle are regarded as directed lines there are eight triangles having the same vertices. As the operations of changing the direction of the sides are independent operations of period two, it follows that each of these triangles may be transformed into all of them by the operations of the group of order 8 which contains seven operations of period two. Similarly, we obtain 64 trihedral angles from a given one by giving direction to the edges and the face angles. The corresponding 64 spherical triangles can be obtained from any one of them by the 64 operations of the group containing 63 operations of period two. These triangles have been studied in details by Study, *Sächsischen Abhandlungen*, Vol. 20, pp. 87-231. A very brief account of them is found in Jones' *Trigonometry*, 1896, p. 168.

The main interest in employing group theory in the study of trigonometry is due to the fact these same groups have extensive application in other fields of mathematics. For instance, by means of the given operations of subtracting from unity and inverting we associate, in general, six expressions with every expression. The six expressions associated with $\sin^2 A/2=\frac{(s-b)(s-c)}{bc}$ are

$$\frac{(s-b)(s-c)}{bc},\ \frac{s(s-a)}{bc},\ \frac{bc}{(s-b)(s-c)},\ \frac{s(a-s)}{(s-b)(s-c)},\ \frac{(s-b)(s-c)}{s(a-s)},\ \frac{bc}{s(s-a)}.$$

These are respectively,

$$\sin^2\frac{A}{2},\quad \cos^2\frac{A}{2},\quad \csc^2\frac{A}{2},\quad -\cot^2\frac{A}{2},\quad -\tan^2\frac{A}{2},\quad \sec^2\frac{A}{2}.$$

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*Subtracting from x_1 and dividing x_1^2 gives rise to this dihedral rotation group for every value of x_1. In the present case $x_1=1$.

No new expressions are obtained by subtracting these from unito or by inverting them. The study of equality among these functions becomes a special case when the group generated by the operations $x_1 - n$, x_1^2/n transforms a point into less than six distinct points. This question has been completely solved for the general dihedral rotation group.

THE EXPRESSION OF THE AREAS OF POLYGONS IN DETERMINANT FORM.

By R. P. BAKER.

The area of the triangle the rectangular coördinates of whose vertices are (x_1, y_1); (x_2, y_2); (x_3, y_3), is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

For the area of a quadrilateral whose vertices are 1, 2, 3, 4, diagonals (13) and (24) and such that circuits (123), (134) have the area on the left, we have

$$\frac{1}{2}\begin{vmatrix} x_1, & y_1, & 1, & 0 \\ x_2, & y_2, & 1, & 1 \\ x_3, & y_3, & 1, & 0 \\ x_4, & y_4, & 1, & 1 \end{vmatrix} \equiv \frac{1}{2}\begin{vmatrix} x_1, & y_1, & 1, & -k \\ x_2, & y_2, & 1, & 1-k \\ x_3, & y_3, & 1, & -k \\ x_4, & y_4, & 1, & 1-k \end{vmatrix}.$$

The case of the pentagon or polygon of more sides than 5 is different.

Suppose that the area P of the pentagon can be expressed by

$$\lambda\begin{vmatrix} x_1, & y_1, & 1, & a_1, & b_1 \\ x_2, & y_2, & 1, & a_2, & b_2 \\ x_3, & y_3, & 1, & a_3, & b_3 \\ x_4, & y_4, & 1, & a_3, & b_4 \\ x_5, & y_5, & 1, & a_5, & b_5 \end{vmatrix} = \lambda\begin{vmatrix} x_1, & y_1, & 1, & 0, & 0 \\ x_2, & y_2, & 1, & a'_2, & b'_2 \\ x_3, & y_3, & 1, & a'_3, & b'_3 \\ x_4, & y_4, & 1, & a'_4, & b'_4 \\ x_5, & y_5, & 1, & a'_5, & b'_5 \end{vmatrix}$$

the latter being obtained from the former by subtracting multiples of columns.

Expanding by Laplace's method in minors of the first three columns we get the area as a sum of multiples of triangular areas all having the point 1 as vertex. The multipliers must obviously be equal. Hence all the determinants $\begin{vmatrix} a'_i & b'_i \\ a'_j & b'_j \end{vmatrix}$ must be equal. This is impossible, for